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A Class of $\mathcal{N}=1$ Supersymmetric RG Flows from Five-dimensional $\mathcal{N}=8$ Supergravity

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We consider the holographic dual of a general class of $\mathcal{N}=1^*$ flows in which all three chiral multiplets have independent masses, and in which the corresponding Yang-Mills scalars can develop particular supersymmetry-preserving vevs. We also allow the gaugino to develop a vev. This leads to a six parameter subspace of the supergravity scalar action, and we show that this is a consistent truncation, and obtain a superpotential that governs the $\mathcal{N} = 1^*$ flows on this subspace. We analyse some of the structure of the superpotential, and check consistency with the asymptotic behaviour near the UV fixed point. We show that the dimensions of the six couplings obey a sum rule all along the $\mathcal{N}=1^*$ flows. We also show how our superpotential describes part of the Coulomb branch of the non-trivial $\mathcal{N} = 1$ fixed point theory.

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1. Introduction

The study of holographic RG flows has become one of the more enduring spin-offs of the AdS/CFT correspondence. In particular the flows of $\mathcal{N}=4$ supersymmetric Yang-Mills theories under "bilinear perturbations[1]" has been extensively studied [1-11] using gauged $\mathcal{N}=8$ supergravity in five dimensions [12,13]. The most extensive studies have been made for supersymmetric flows because these are generically simpler, more controllable, and sometimes enable direct comparisons of the supergravity and field theory limits. Thus far the flows have usually been the simpler ones with a higher degree of global symmetry, [2-6] though a relatively recent paper [11] has pushed this restriction back even further.

In this letter we will give the supergravity description of an even more general class of flows of $\mathcal{N}=4$ supersymmetric Yang-Mills: $\mathcal{N}=1$ supersymmetric flows with six independent parameters, which may be interpreted as three independent masses of the chiral multiplets, two independent vevs of the scalar fields in these chiral mutiplets, and the vev of the gaugino condensate. We work with gauged $\mathcal{N}=8$ supergravity in five dimensions, and obtain a superpotential describing these flows via steepest descent. Indeed we find four different superpotentials that are trivially related by a $\mathbb{Z}_4$ symmetry: Each of the four superpotentials merely corresponds to a different fermion being identified as the gaugino.

While we have the supergravity description of flows with all three chiral multiplet masses running independently, this is not the most general class of $\mathcal{N}=1$ supersymmetric flow under "bilinear perturbations": It is still possible to flow the scalar and fermion bilinear vevs in more exotic ways than those considered here. On the other hand, we show that the six parameter truncation that we consider here is a consistent truncation of the supergravity model, and thus the flows involve a closed family of bilinear operators. We anticipate that our results will prove valuable in further probing holographic renormalization group flows. In this letter we will content ourselves with exhibiting the superpotential, showing that there are no new critical points, obtaining a sum-rule for the anomalous dimensions of fermion bilinears, and exhibiting the Coulomb branch of the $\mathcal{N}=1$ supersymmetric Leigh-Strassler point [14].

[1] By this we mean either the introduction of mass terms for fundamental fields, or the turning on of vevs for bilinear operators.

2. The scalar manifold

2.1. A non-compact Cartan sub-algebra

The forty-two scalars of $\mathcal{N} = 8$ supergravity parametrize non-compact coset space $E_{6(6)}/USp(8)$, and can be described by a 27×27 matrix, $\mathcal{V}_{AB}{}^{ab}$. Working in the $SL(6,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$ basis, an element of the $E_{6(6)}$ Lie algebra can be written in the block form as [12]

$$\mathcal{X} \;=\; \begin{pmatrix} -4\,\Lambda^{[M}{}_{[I}\delta^{N]}{}_{J]} & \sqrt{2}\,\Sigma_{IJP\beta} \\ \sqrt{2}\,\Sigma^{MNK\alpha} & \Lambda_P^K\,\delta^\alpha{}_\beta \;+\; \Lambda_\beta^\alpha\,\delta^K{}_P \end{pmatrix}\;, \tag{2.1}$$

where Λ_P^K, Λ_β^α represent elements of the $SL(6,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$ Lie algebra and $\Sigma_{IJP\beta}$ transforms in the $(\mathbf{20},\mathbf{2})$ of $SL(6,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$. The scalar fields to which we wish to restrict are most easily characterized as follows. Introduce Cartesian coordinates $x^I, I = 1,\ldots,6$, and $y^\alpha, \alpha = 1,2$, and define the differential form Σ by:

$$\Sigma \;=\; \frac{1}{12}\,\Sigma_{IJK\alpha}\,dx^I \,\wedge\, dx^J \,\wedge\, dx^K \,\wedge\, dy^\alpha\;. \tag{2.2}$$

Now introduce complex coordinates $z_1 = x^1 + ix^2$, $z_2 = x^3 - ix^4$, $z_3 = x^5 - ix^6$, and $z_4 = y^1 + iy^2$. Four of the generators we seek can be defined by setting:

$$\Sigma \;=\; k\sum_{i=1}^{4}\varphi_i\;(\;\Upsilon_i \;+\; \overline{\Upsilon_i}\;)\,, \tag{2.3}$$

where k is a normalization constant and

$$\begin{aligned} \Upsilon_1 \;&\equiv\; -\,dz_1\wedge dz_2\wedge dz_3\wedge dz_4\,, \qquad & \Upsilon_2 \;&\equiv\; -d\overline{z}_1\wedge d\overline{z}_2\wedge dz_3\wedge dz_4 \\ \Upsilon_3 \;&\equiv\; -\,d\overline{z}_1\wedge dz_2\wedge d\overline{z}_3\wedge dz_4\,, \qquad & \Upsilon_4 \;&\equiv\; -dz_1\wedge d\overline{z}_2\wedge d\overline{z}_3\wedge dz_4\,. \end{aligned} \tag{2.4}$$

The remaining two generators are given by taking the $SL(6,\mathrm{I\!R})$ Lie algebra element to be of the form:

$$\Lambda \;\equiv\; \mathrm{diag}(\alpha+\beta,\alpha+\beta,\alpha-\beta,\alpha-\beta,-2\,\alpha,-2\,\alpha,)\,. \tag{2.5}$$

One can easily verify that the foreging six generators constitute a (non-compact) Cartan sub-algebra of $E_{6(6)}$, and indeed one can show that they constitute the diagonal elements of another $SL(6,\mathrm{I\!R})\times SL(2,\mathrm{I\!R})$ sub-algebra (distinct from the one described above).

The kinetic term for these scalars is:

$$-3\,(\partial\alpha)^2 \;-\; (\partial\beta)^2 \;-\; \tfrac{1}{2}\sum_{j=1}^{4}(\partial\varphi_j)^2\,, \tag{2.6}$$

where the normalization constant k in (2.3) has been chosen so as to canonically normalize the kinetic terms of the φ_j.

In terms of the theory on the brane, the parameters φ_i are dual to the operators:

$$\mathrm{Tr}(\lambda_i\lambda_i) \, + \, h.c. \, ,$$

while α and β are, respectively, dual to

$$\begin{aligned} \mathcal{O}_1 \; &\equiv \mathrm{Tr}(X_1^2 + X_2^2 + X_3^2 + X_4^2 - 2\, X_5^2 - 2\, X_6^2) \quad \text{and} \\ \mathcal{O}_2 \; &\equiv \mathrm{Tr}(X_1^1 + X_2^2 - X_3^2 - X_4^2) \, . \end{aligned} \tag{2.7}$$

It should also be remembered that the operator:

$$\mathcal{O}_0 \;\equiv\; \mathrm{Tr}\Big(\sum_{i=1}^{6} X_i^2\Big) \, , \tag{2.8}$$

has no supergravity dual in the gauged $\mathcal{N}=8$ supergravity theory, but that the field theory on the brane always adds an appropriate amount of $\mathcal{O}_0$ to the operators $\mathcal{O}_j, j=1,2$ so as to preserve supersymmetry and positivity.

2.2. Consistency of the truncation

The simplest way to establish the consistency of a truncation is to exhibit a symmetry (continuous or discrete) of the action for which the truncated sector consists of precisely all the singlet fields under that symmetry. It then follows from Schur's lemma that all variations of the action must be at least quadratic in non-singlet fields, and hence it is consistent with the equations of motion to set all non-singlets fields to zero. We now exhibit a symmetry that reduces the forty-two scalars to the six described above.

Consider the following two $\mathbb{Z}_2$ generators in $SO(6)$:

$$diag(-1,-1,-1,-1,1,1) \quad \text{and} \quad diag(1,1,-1,-1,-1,-1) \, . \tag{2.9}$$

This $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be viewed as simultaneously negating pairs of the complex coordinates z_i. As a result, demanding invariance under these symmetries of the potential means that

Λ^I_J must be block diagonal with three 2×2 blocks. Similarly, it requires that Σ contain one of each pair $\{dz_i, d\bar{z}_i\}, i = 1, \ldots, 3$. Now introduce the matrices:

$$\mathcal{J}_1 \ = \ \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix} , \qquad\qquad \mathcal{J}_2 \ = \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

considered as elements of $SO(6)\times SO(2) \subset SL(6,\mathbb{R})\times SL(2,\mathbb{R})$. The potential is invariant under the action of both of these matrices. Consider the symmetry group, $\mathcal{G}$, generated by the $\mathbb{Z}_2\times\mathbb{Z}_2$ of (2.9), along with the combined action $\mathcal{J}_1\mathcal{J}_2$. The latter generates a $\mathbb{Z}_4$, and invariance requires that the 2×2 blocks in Λ^I_J are in fact independent multiples of the 2×2 identity matrix. Remembering that Λ^I_J is traceless, we have thus reduced it to (2.5). The combined action of $\mathcal{J}_1\mathcal{J}_2$ rotates $z_a \to -iz_a, a = 1, \ldots, 4$, and invariance under this (and the other $\mathbb{Z}_2$'s) reduces Σ to a linear combination of the Υ_i and their complex conjugates (2.4). We now observe that Λ in (2.5) commutes with an $SO(2)\times SO(2)\times SO(2) \subset SO(6)$, and that these $U(1)$'s, along with the $U(1) \subset SL(2,\mathbb{R})$ can be used to reduce to the real linear combination in (2.3). Putting it another way, the discrete symmetry reduces the the $E_{6(6)}/Usp(8)$ coset to

$$\mathcal{M} \ = \ \left(\frac{SU(1,1)}{U(1)}\right)^4 \ \times \ SO(1,1) \ \times \ SO(1,1)\,, \tag{2.10}$$

where the denominator $U(1)^4 \subset SO(6) \times SO(2)$, the numerator $SU(1,1)$'s are parametrized by the Υ_a, and the $SO(1,1)$'s are parametrized by α and β. The $U(1)$'s can then be used to gauge fix the complex parameter in each $SU(1,1)$ to be real.

3. The supergravity superpotential and supersymmetric flows

3.1. The superpotential

The superpotential is generically extracted as an eigenvalue of the tensor, W_{ab}, of the five-dimensional $\mathcal{N} = 8$ theory. For the scalar manifold that we consider here, the eigenvectors are all constant vectors, and indeed are relatively simple:

$$\begin{aligned} \vec{v}_1 \ &= (-1,0,1,0,0,1,0,1)\,, \qquad & \vec{v}_2 \ &= \ (0,1,0,1,1,0,-1,0)\,, \\ \vec{v}_3 \ &= (1,0,1,0,0,-1,0,1)\,, \qquad & \vec{v}_4 \ &= \ (0,1,0,-1,1,0,1,0)\,, \\ \vec{v}_5 \ &= (1,0,-1,0,0,1,0,1)\,, \qquad & \vec{v}_6 \ &= \ (0,1,0,1,-1,0,1,0)\,, \\ \vec{v}_7 \ &= (1,0,1,0,0,1,0,-1)\,, \qquad & \vec{v}_8 \ &= \ (0,-1,0,1,1,0,1,0)\,. \end{aligned} \tag{3.1}$$

There are four distinct eigenvalues corresponding to the spaces spanned by $\vec{v}_{2a-1}, \vec{v}_{2a}$, for $a = 1, \ldots, 4$. The first eigenvalue is:

$$\begin{aligned} \mathcal{W} \;=\; \frac{1}{4}\Big[& \big(\rho^{-4} - \rho^2\,(\nu^2 + \nu^{-2})\big)\,\cosh(2\,\varphi_1) \;+\; \big(-\rho^{-4} + \rho^2\,(-\nu^2 + \nu^{-2})\big)\,\cosh(2\,\varphi_2) \;+ \\ & \big(-\rho^{-4} + \rho^2\,(\nu^2 - \nu^{-2})\big)\,\cosh(2\,\varphi_3) \;-\; \big(\rho^{-4} + \rho^2\,(\nu^2 + \nu^{-2})\big)\,\cosh(2\,\varphi_4)\Big]\,, \end{aligned} \tag{3.2}$$

where $\rho \equiv e^{\alpha}$ and $\nu \equiv e^{\beta}$. The remaining three eigenvalues are obtained from this by doing the pairwise permutations:

$$\varphi_1 \leftrightarrow \varphi_4\,,\varphi_2 \leftrightarrow \varphi_3\,; \qquad \varphi_1 \leftrightarrow \varphi_3\,,\ \varphi_2 \leftrightarrow \varphi_4\,; \quad \text{and} \quad \varphi_1 \leftrightarrow \varphi_2\,,\ \varphi_3 \leftrightarrow \varphi_4\,. \tag{3.3}$$

As one would hope, $\mathcal{W}$ does indeed provide a superpotential in that the supergravity potential, $\mathcal{P}$, is given by:

$$\mathcal{P} \;=\; \frac{1}{8}\sum_{i=1}^{4}\Big(\frac{\partial\mathcal{W}}{\partial\varphi_i}\Big)^2 \;+\; \frac{1}{48}\Big(\frac{\partial\mathcal{W}}{\partial\alpha}\Big)^2 \;+\; \frac{1}{16}\Big(\frac{\partial\mathcal{W}}{\partial\beta}\Big)^2 \;-\; \frac{1}{3}\,\mathcal{W}^2\,. \tag{3.4}$$

Indeed, because the actions (3.3) interchange the eigenvalues, it follows that (3.4) yields the supergravity potential, $\mathcal{P}$, if $\mathcal{W}$ is chosen to be any of the eigenvalues of W_{ab}.

One should also note that $\mathcal{W}$ is *invariant, up to a sign,* under the permutation group S_3. These permutations are generated by the transformations:

$$\begin{aligned} p_1 \;&:\quad \varphi_1 \leftrightarrow \varphi_2\,, \quad \varphi_3 \rightarrow \varphi_3\,, \quad \alpha \rightarrow -\tfrac{1}{2}(\alpha+\beta)\,, \quad \beta \rightarrow \tfrac{1}{2}(\beta - 3\,\alpha) \\ p_2 \;&:\quad \varphi_2 \leftrightarrow \varphi_3\,, \quad \varphi_1 \rightarrow \varphi_1\,, \quad \alpha \rightarrow \alpha\,, \quad \beta \rightarrow -\beta\,, \end{aligned} \tag{3.5}$$

and these act on $\mathcal{W}$ according to: $p_1 : \mathcal{W} \rightarrow -\mathcal{W}$ and $p_2 : \mathcal{W} \rightarrow \mathcal{W}$.

In terms of the physics on the brane, the choice of the eigenvalue of W_{ab} corresponds to the choice of which of the four fermions will be the gaugino. The permutation symmetry generated by (3.5) represents the permutations of the three chiral multiplets. Thus the physical content of all four possible superpotentials is the same, and we therefore stay with the choice (3.2).

3.2. Supersymmetric flows

As is common when considering supergravity descriptions of RG flows, we will take the five-dimensional metric to have the form

$$ds^2_{1,4} = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu - dr^2 \,. \tag{3.6}$$

For a supersymmetric flow one requires that the variations of the spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ fields to vanish. The former leads to the condition:

$$\frac{dA}{dr} \;=\; -\frac{g}{3}\,W \;. \tag{3.7}$$

The variations of the spin-$\frac{1}{2}$ fields lead to the equations:

$$\begin{aligned} \frac{d\alpha}{dr} \;&=\; \frac{g}{12}\,\frac{\partial W}{\partial \alpha}\,, \qquad\quad \frac{d\beta}{dr} \;=\; \frac{g}{4}\,\frac{\partial W}{\partial \beta}\,, \\ \frac{d\varphi_j}{dr} \;&=\; \frac{g}{2}\,\frac{\partial W}{\partial \varphi_j}\,, \qquad\quad j = 1, \ldots, 4 \;. \end{aligned} \tag{3.8}$$

For later convenience we introduce the canonically normalized scalars $\varphi_5 \equiv \sqrt{6}\,\alpha$, $\varphi_6 \equiv \sqrt{2}\,\beta$.

It is straightforward, but rather tedious to verify directly that these equations imply that all the spin-$\frac{1}{2}$ variations vanish. We have indeed confirmed this using *Mathematica*^{TM}, however there is a rather general analytic argument that establishes the same result.

Suppose that ξ^a is an eigenvector of W_{ab}, with eigenvalue $\mathcal{W}$. Let $\zeta^a \equiv \Omega^{ab}(\xi^b)^*$, where Ω is the symplectic form defined in [12]. From the symplectic reality condition satisfied by W_{ab} it follows that ζ^a must be an eigenvector with eigenvalue $\mathcal{W}^*$. For simplicity, assume that ξ, ζ and $\mathcal{W}$ are real (as they are here).

To prove that the spin-$\frac{1}{2}$ variations vanish we must show that the following expression vanishes:

$$P_{0\,abcd}\,\xi^d \;\pm\; \frac{1}{2}\,g\,A_{dabc}\,\zeta^d \;, \tag{3.9}$$

for one uniform choice of the sign. The quantities in this equation are defined in [12]. We consider the modulus squared of (3.9). Recalling that the canonically normalized kinetic term is $\frac{1}{24}|P_{\mu\,abcd}|^2$, it is relatively easy to show that:

$$\frac{1}{24}\,P_{0\,abcd}P_0{}^{abce}\,\xi^d\,\xi^e \;=\; \frac{1}{16}\sum_{j=1}^{6}(\partial\varphi_j)^2 \;. \tag{3.10}$$

The factor of $\frac{1}{8}$ compared to the usual kinetic term arises from the fact that we are not taking the trace over the last index, but we simply contract with ξ.

The cross-terms between $P_{0\,abcd}$ and A_{dabc} can be simplified using the identity (3.21) of [12]:

$$D_\mu W_{ab} \;=\; \frac{2}{3}\, P_{\mu\,(a}{}^{cde}\, A_{b)cde}\;. \tag{3.11}$$

Finally, contracting the following identity ((3.31) of [12]):

$$\frac{1}{24}\, W^{ac}\, W_{cb} \;-\; \frac{1}{96}\, A^{acde}\, A_{bcde} \;=\; -\frac{1}{8g^2}\, \mathcal{P}\, \delta^a_b\;, \tag{3.12}$$

with ξ^b, and using equation (3.4), it follows immediately that:

$$A^{acde}\, A_{bcde}\, \xi^b \;=\; \frac{3}{2}\left(\sum_{i=1}^{6} \frac{\partial \mathcal{W}}{\partial \varphi_i}\right) \xi^a\;. \tag{3.13}$$

Putting together (3.10), (3.11) and (3.13), we arrive at:

$$\Big| P_{0\,abcd}\, \xi^d \;\pm\; \frac{1}{2}\, g\, A_{dabc}\, \zeta^d \Big|^2 \;=\; \frac{3}{2}\, |\xi^a|^2 \sum_{i=1}^{6} \Big| \frac{d\varphi_i}{dr} \;\pm\; \frac{1}{2}\, g\, \frac{\partial \mathcal{W}}{\partial \varphi_i} \Big|^2 . \tag{3.14}$$

It therefore follows that the spin-$\frac{1}{2}$ variations vanish if and only if the steepest descent equations in (3.8) are satisfied.

4. Properties of the Superpotential

4.1. Behaviour near the maximally supersymmetric phase

In the maximally supersymmetric phase all the supergravity scalars vanish. In the neighbourhood of this point we find:

$$\begin{aligned} \mathcal{W} \;\sim\; & -\frac{3}{2} \;-\; \frac{1}{2}\,(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + 3\,\varphi_4^2 + 2\,(\varphi_5^2 + \varphi_6^2)) \\ & -\; \sqrt{\frac{2}{3}}\; \varphi_5\,(2\,\varphi_1^2 - \varphi_2^2 - \varphi_3^2) \;-\; \sqrt{2}\,\varphi_6\,(\varphi_2^2 - \varphi_3^2)\;, \end{aligned} \tag{4.1}$$

where, one should recall, that $\varphi_5 \equiv \sqrt{6}\,\alpha$, $\varphi_6 \equiv \sqrt{2}\,\beta$ are the canonically normalized supergravity scalars.

The quadratic terms imply that perturbations in φ_1, φ_2 and φ_3 are non-normalizable in AdS_5, and so represent non-trivial masses for the corresponding fermion fields. The quadratic terms in $\varphi_j, j = 5, 6$ give rise to normalizable AdS_5 modes, however, as in [4], the cubic mixing terms between $\varphi_{5,6}$ and $\varphi_j^2, j = 1, 2, 3$ imply that if the modes $\varphi_j, j = 1, 2, 3$ are excited then non-normalizable modes are excited for $\varphi_j, j = 5, 6$. Indeed, this must happen because of $\mathcal{N} = 1$ supersymmetry: turning on a fermion means that an entire chiral superfield is becoming massive, and so the boson masses must be turned on in precisely the proper manner.

It is not difficult to see that the cubic terms in (4.1) do this correctly. First, if φ_1 runs, then only φ_5 flows in a non-normalizable manner: and this is dual to $\mathcal{O}_1 - \mathcal{O}_0$ [2]. This means that the superfield Φ_3 is developing a mass. If φ_2 runs, then, at lowest order we have $\frac{d\alpha}{dr} \sim \frac{1}{3}\varphi_2^2$ and $\frac{d\beta}{dr} \sim -\varphi_2^2$, *i.e.* to lowest order $\beta = -3\alpha$. From (2.5) the corresponding $SL(6, \mathbb{R})$ scalar is $\Lambda = 2\alpha\, \mathrm{diag}(2, 2, -1, -1, -1, -1)$, which means that the superfield Φ_1 is developing a mass. Similarly a flow of φ_3 means that to lowest order $\beta = +3\alpha$, and Φ_2 is developing a mass.

4.2. Non-trivial critical points

The non-trivial, supersymmetric critical point discovered in [1] corresponds to setting:

$$\alpha = -\frac{1}{6}\log(2) \ ; \quad \beta = 0 \ ; \quad \varphi_1 = \frac{1}{2}\log(3) \ ; \quad \varphi_j = 0 \ , \ j = 2, 3, 4 \ . \tag{4.2}$$

A careful analysis shows that there are six non-trivial critical points of the superpotential (3.2), but that they are all equivalent, and related to (4.2) via the permutation symmetry (3.5). In terms of the physics on the brane, they correspond to the choice of superfield that is to be integrated out to obtain the new phase of the theory [4]. The absence of other non-trivial critical points in the superpotential is consistent with the absence of other non-trivial IR fixed points of $\mathcal{N} = 4$ Yang-Mills theory.

4.3. The Coulomb branch of the Leigh-Strassler point, and other truncations

There are several restrictions of our superpotential that are consistent with the equations of motion. For example, the φ_j only enter into the superpotential via $\cosh(2\varphi_j)$ and hence $\frac{\partial \mathcal{W}}{\partial \varphi_j}$ is proportional to $\sinh(2\varphi_j)$. This means that it is always consistent to set any subset of the φ_j to zero.

More generally, suppose we set $\varphi_2 = \pm\varphi_3$. Then, for consistency we must have $\frac{\partial \mathcal{W}}{\partial \varphi_2} = \frac{\partial \mathcal{W}}{\partial \varphi_3}$. This can only be satisfied if either $\varphi_2 = \varphi_3 = 0$, or if $\beta \equiv 0$. One can then

[2] As explained earlier, we are always free at add an appropriate amount of the operator $\mathcal{O}_0$.

check that indeed $\frac{\partial \mathcal{W}}{\partial \beta} = 0$ when one has $\beta = 0$ and $\varphi_2 = \varphi_3$, and so this is indeed a consistent truncation. (Conversely, one can easily show that it is consistent to set $\beta = 0$ if and only if $\varphi_2 = \pm\varphi_3$.) The superpontential reduces to:

$$\widetilde{\mathcal{W}} \;=\; -\frac{1}{4\,\rho^4}\,\big(\cosh(2\varphi_1) - 2\,\cosh(2\varphi_2) - \cosh(2\varphi_4)\big) \;-\; \frac{1}{2}\,\rho^2\;\big(\cosh(2\varphi_1) + \cosh(2\varphi_4)\big). \tag{4.3}$$

If one sets $\varphi_4 = 0$, then this superpotential reduces to that considered [11]. If one further sets $\varphi_2 = 0$ then one gets the superpotential of [4]. Alternatively, if one sets $\varphi_1 = \varphi_4 = 0$, one gets the superpotential considered in [4]. (Note that the parameter we call α here is the negative of that used in [4,11].)

Another interesting class of truncations are those probing the Coulomb branch around the massive $\mathcal{N}=1$ flow considered in [4]. That is, set all the φ_j to zero, except φ_1. One then has the superpotential:

$$\widehat{\mathcal{W}} \;=\; \frac{1}{4\,\rho^4}\,\big(\cosh(2\varphi_1) - 3\big) \;-\; \frac{1}{4}\,\rho^2\,(\nu^2 + \nu^{-2})\,\big(\cosh(2\varphi_1) + 1\big)\,. \tag{4.4}$$

Setting $\varphi_2 = \varphi_3 = 0$ in (4.1), we see that φ_6 no longer mixes linearly with other fields, and so now φ_6 corresponds to an AdS_5 normalizable mode. As explained in [5], φ_6 represents a modulus of the Coulomb branch of the $\mathcal{N}=4$ theory. In the neighbourhood of the non-trivial critical point the scaling dimension of φ_6 is also 2, and so from the tables in [4] we can identify φ_6 as being dual to one of the operators $Tr(\bar{\Phi}T^A\Phi)$, where T^A are generators of an $SU(2)$ subgroup of the $SO(6)$ R-symmetry. Since $\varphi_6 = \sqrt{2}\,\beta$ commutes with $U(1)^3 \subset SO(6)$, it follows that, at the non-trivial critical point, φ_6 must be dual to $Tr(\bar{\Phi}T^3\Phi) \sim \mathrm{Tr}(X_1^1 + X_2^2 - X_3^2 - X_4^2) \equiv \mathcal{O}_2$ of (2.7). In other words φ_6 is dual to exactly the same operator at both ends of the flow, suggesting that $\mathcal{O}_2$ does not mix with other operators along the flow. One can also verify that, all along the flow, the hessian of $\mathcal{W}$ has a constant eigenvector corresponding to fluctuations in the φ_6-direction, which means that φ_6 does not mix with other fields in supergravity. We therefore conclude that φ_6 represents a Coulomb modulus for the complete flow.

We have looked at the flows from the non-trivial critical point. As is common, we will normalize the coordinate, r, by fixing $g = 2$. To lowest order, the flows in β correspond to vevs in $\mathrm{Tr}(\bar{\Phi}_1\Phi_1)$ or $\mathrm{Tr}(\bar{\Phi}_2\Phi_2)$. Generically for $\alpha > \alpha_0 \equiv -\frac{1}{6}\log(2)$ one has flows with asymptotic behaviour:

$$\alpha \sim -\tfrac{1}{20}\,\log(\tfrac{5}{3}\,r)\,, \quad \beta \sim \pm 3\,\alpha\,, \quad \varphi_1 \sim 6\,\alpha\,, \quad A \sim -2\,\alpha \sim \tfrac{1}{10}\,\log(\tfrac{5}{3}\,r)\,, \tag{4.5}$$

as $r \to 0$. We have chosen a constant of integration so as to put the infra-red singularity at $r = 0$. For $\alpha < \alpha_0$ one has flows with asymptotics:

$$\varphi_1 \to a\, r^{\frac{3}{4}} \to 0\,, \quad \alpha \sim \tfrac{1}{4}\, \log\left(\tfrac{4}{3}\, r\right), \quad \beta \to \beta_0\,, \quad A \sim \tfrac{1}{4}\, \log(r)\,, \tag{4.6}$$

where a and β_0 are constants.[3]

Between these two classes of flows there are two natural ridge lines that asymptote to lines with $\beta = \pm 3\alpha$ as $\alpha \to -\infty$. To get the flows along these ridges requires a rather delicate asymptotic analysis, and one finds that the value of φ_1 relaxes to zero *very slowly* along the asymptotes. Again choosing the constant of integration to put the infra-red singularity at $r = 0$, we find:

$$\begin{aligned} \alpha \ &\sim\ \frac{1}{4}\, \log(\tfrac{2}{3}\, r)\,, \quad \beta \ \sim\ \pm(3\alpha + \varphi_1^2)\,, \\ \varphi_1^2 \ &\sim\ \frac{1}{a - 6\, \log(r)}\,, \qquad A(r) \ \sim\ \log(r)\,. \end{aligned} \tag{4.7}$$

for some constant of integration a. While this suggests that the flow is returning to the Coulomb branch of the $\mathcal{N} = 4$ theory in the infra-red, it is doing it much more slowly than in (4.6). This may indicate that some intrisically different physics is to be associated with these ridge-line flows. Note that the asymptotic behaviour of e^{2A} is also significantly different from (4.6): Indeed, the space-time metric is now asymptotic to:

$$ds_{1,4}^2 \ \sim\ -dr^2 \ +\ r^2\, \eta_{\mu\nu} dx^\mu dx^\nu\,. \tag{4.8}$$

The scale on the brane goes linearly with r. Amusingly enough, the asymptotic behaviour of φ_1^2 is somewhat reminiscent of a running gauge coupling.

It was argued in [8] that the criterion for separating the physical from the unphysical flows was to require that the supergravity potential, $\mathcal{P}$, remain bounded above along the flows. This criterion excludes the flows (4.5) in which $\alpha \to +\infty$, and suggests that the flows with $\alpha \to -\infty$ are physical, including the ridge-line flows. In [8] the $\mathcal{N} = 1$ flows with $\beta = 0$ were considered, and it was further argued that the flows with $\alpha \to +\infty$ (and $\varphi_1 \neq 0$) were unphysical because they corresponded to giving $\mathcal{O}_1 + 2\mathcal{O}_0$ a vev or a mass of the wrong sign, violating positivity. It seems reasonable to assume that something similar is happening here, and that the ridge-line flows correspond to the pure Coulomb branch of the Leigh-Strassler theory, while the other flows with $\alpha \to -\infty$ correspond to following

[3] There are other classes of flows in the region in which $\cosh(2\varphi_1) > 3$, but these are not accesible from the non-trivial critical point.

a relevant perturbation away from the conformal point. This implies that the ridge line flow would be given by setting the initial velocities as follows:

$$\frac{d\,\alpha}{d\,r}\bigg|_{r\to\infty} = \mathcal{O}(e^{-a\,r})\,, \qquad \frac{d\,\beta}{d\,r}\bigg|_{r\to\infty} \sim b\,e^{-2\,r}\,, \qquad \frac{d\,\varphi_1}{d\,r}\bigg|_{r\to\infty} = \mathcal{O}(e^{-a\,r})\,, \tag{4.9}$$

for some constant b, and for some constant a that is larger than the largest scaling dimension in the (α, φ_1)-space at the non-trivial critcal point. Unfortunately we do not know the analytic solution. However, the expansion of the superpotential near the critical point has terms $\alpha\beta^2$ and $\varphi_1\beta^2$, which means that the velocities of α and φ_1 must actually be of order $b^2 e^{-4r}$. Numerical analysis is consistent with this: We find that to get the ridge-line flows the initial velocities of α and φ_1 vanish as the square of the initial velocity of β in the limit $r \to \infty$.

We therefore conclude that the two ridges are the Coulomb branch of the Leigh-Strassler theory, with one ridge representing a non-zero vev for $\mathrm{Tr}(\bar{\Phi}_1\Phi_1)$ and the other a non-zero vev for $\mathrm{Tr}(\bar{\Phi}_2\Phi_2)$.

4.4. A supersymmetric sum rule

It is elementary to show that the superpotential (3.2) satisfies the following two identities:

$$\sum_{i=1}^{4} \frac{\partial^2 \mathcal{W}}{\partial \varphi_j^2} = 4\,\mathcal{W}\,, \qquad \sum_{i=5}^{6} \frac{\partial^2 \mathcal{W}}{\partial \varphi_j^2} \equiv \frac{1}{6}\,\frac{\partial^2 \mathcal{W}}{\partial \alpha^2} + \frac{1}{2}\,\frac{\partial^2 \mathcal{W}}{\partial \beta^2} = \frac{8}{3}\,\mathcal{W}\,. \tag{4.10}$$

Consider a general supersymmetric flow:

$$\frac{d\varphi_j}{dr} = \frac{g}{2}\,\frac{\partial W}{\partial \varphi_j}\,, \qquad j = 1, \ldots, 6\,, \tag{4.11}$$

and suppose that $\delta\varphi_j$ is some small deviation away from this flow. Then to lowest order:

$$\frac{d}{dr}\,\delta\varphi_j = \frac{g}{2}\sum_{k=1}^{6} \frac{\partial^2 W}{\partial\varphi_j \partial\varphi_k}\,\delta\varphi_k\,. \tag{4.12}$$

Now recall that the metric (3.6) implies that the scale on the brane is $\mu \equiv e^{A(r)}$, and so $\mu\frac{d}{d\mu} = \frac{d}{dA}$. So we change variables to A using (3.7) to obtain:

$$\mu\frac{d}{d\mu}\,\delta\varphi_j \equiv \frac{d}{dA}\,\delta\varphi_j = -\left[\frac{3}{2\mathcal{W}}\sum_{k=1}^{6} \frac{\partial^2 W}{\partial\varphi_j \partial\varphi_k}\right]\delta\varphi_k\,. \tag{4.13}$$

The matrix in square brackets is real and symmetric, and is thus diagonalizable. Its eigenvalues represent the scaling dimensions of the couplings spanned by $\delta\varphi_j$, and sum of these scaling dimensions is given by the trace of this matrix. It follows from this and from (4.10) that sum of the scaling dimensions of the fields spanned by $\delta\varphi_j$ is simply: $\frac{3}{2} \times (4 + \frac{8}{3}) = 10$, and this is true all along the flow, and not just at critical points.

One can, of course, verify this sum rule at the two known critical points. At the $\mathcal{N} = 4$ point the scaling dimensions of the φ_i are $(1, 1, 1, 3, 2, 2)$, and at the non-trivial critical point the φ_i are not the diagonal basis, but when one diagonalizes one finds eigenvalues: $(\frac{3}{2}, \frac{3}{2}, 2, 3, 1 + \sqrt{7}, 1 - \sqrt{7})$. In both cases the sum is 10.

5. Final Remarks

In this paper we have constructed a superpotential, $\mathcal{W}$, that describes, in five dimensions, the broadest class of $\mathcal{N} = 1$ supersymmetric flows so far obtained for relevant perturbations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In particular, our superpotential contains, as special *consistent* truncations, all the superpotentials considered previously. We have shown that there are no physically new critical points of the superpotential, $\mathcal{W}$, and thus, as one would expect from field theory, there are no new non-trivial supersymmetric fixed points. On the other hand, while we have not exhibited them here, the potential, $\mathcal{P}$, does appear to have some new non-trivial (non-supersymmetric) critical points. The more general superpotential has enabled us to obtain a sum rule for anomalous dimensions, and to probe the Coulomb branch of the non-trivial $\mathcal{N} = 1$ fixed point theory of [14].

It would, of course be interesting to obtain the full ten-dimensional versions of these flows, but the task may be computationally unfeasible. It is elementary to use the results of [1,15] to compute the metric and dilaton backgrounds. The latter is relatively simple. The complexity comes not only from the metric, but also from the unknown tensor gauge field backgrounds.

More practically, one might hope to obtain a better understanding of the ten-dimensional origins of the sum rule, or an understanding of the geometric structure of the one parameter flow along the Coulomb branch of the $\mathcal{N} = 1$ fixed point theory considered in section 4.3. We are currently working on this. One interesting, and still somewhat surprising feature, is that if one computes the general dilaton background for our superpotential, and then seeks out the scalar submanifold for which the dilaton/axion background is *trivial* then one is led to precisely the three scalars: α, β, φ_1 with superpotential (4.4).

Thus, this Coulomb branch flow has trivial dilaton/axion and it would be particularly interesting to see the ten-dimensional geometric distinction between the two classes of flows (4.6) and (4.7).

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